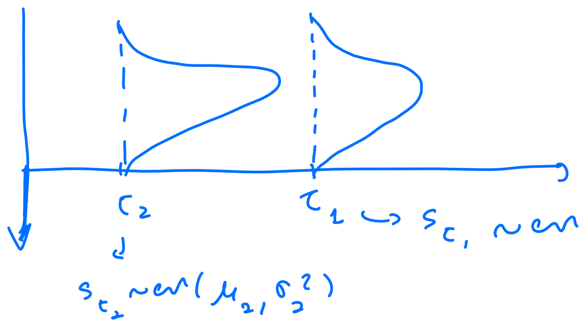
Summary: Martingales
Exercises

Conditional expectation

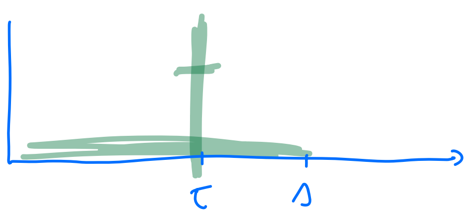
$\{S_t,\ t \geq 0\}$; $\underbrace{\{\mathcal{F}_t\}_{t\geq 0}}_{\text{general filtration}}$; $\underbrace{\{\mathcal{F}_t^S\}_{t\geq 0}}_{\text{natural filtration}}$

$E[S_t \mid \mathcal{F}_s^S] \rightarrow$ expected value of S_t, given information until time s



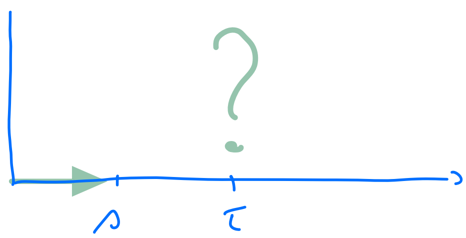
$S_t \sim N(\mu_t, \sigma_t^2)$

if $s \geq t$: $E[S_t \mid \mathcal{F}_s^S] = S_t$ (meaning that in this case S_t is known and no longer a random variable)



$E[S_t \mid \mathcal{F}_t^S] = S_t$

if $s < t$: $E[S_t \mid \mathcal{F}_s^S] = ?$



properties :

(basic) $E[a S_t \mid \mathcal{F}_s^S] = a\, E(S_t \mid \mathcal{F}_s^S)$

$E[a S_t + S_u \mid \mathcal{F}_s^S] = a E[S_t \mid \mathcal{F}_s^S] + E[S_u \mid \mathcal{F}_s^S]$

$E[a \mid \mathcal{F}_s^S] = a$

($s<t$) (more interesting) $E[S_s S_t \mid \mathcal{F}_s^S] = S_s E[S_t \mid \mathcal{F}_s^S]$

$E[S_s \mid \mathcal{F}_s^S] = S_s$

$E[A_t S_t \mid \mathcal{F}_s^S] = E[A_t] E(S_t \mid \mathcal{F}_s^S)$

(with A_t completely independent of $\mathcal{F}_s^S$)

$s_1 \le s_2 < t$ (Tower property) $E\Big[\underbrace{E[S_t \mid \mathcal{F}_{s_1}^S]}_{\text{This is a random variable, that depends on } S_{s_1}} \Big| \mathcal{F}_{s_2}^S\Big] =$

$\Downarrow$

$\mathcal{F}_{s_1}^S \le \mathcal{F}_{s_2}^S$

$= E[S_t \mid \mathcal{F}_{s_1}^S]$

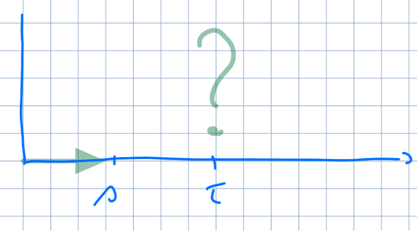
$= E\big[E(S_t \mid \mathcal{F}_{s_2}^S) \mid \mathcal{F}_{s_1}^S\big]$

s_2 $E[S_t \mid \mathcal{F}_{s_2}^S] =$ This is a random variable, that depends on s_2 $\equiv Y_{s_2}$

$E[Y_{s_2} \mid \mathcal{F}_{s_1}^S] =$ This is a random variable that depends on s_1

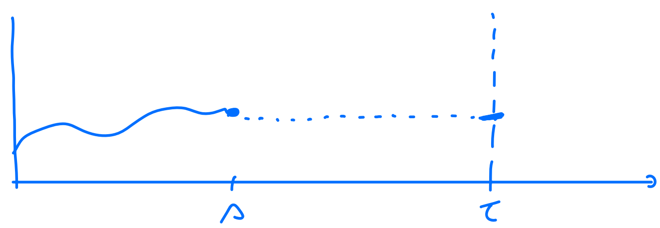
Martingales

if $s < t$: $E[S_t \mid \mathcal{F}^S_s] = ?$ → This is a random variable, that depends on $\mathcal{F}^S_s$, Y_s



Definition: Given a stochastic process $\{S_t, t \geq 0\}$ and a filtration $\{\mathcal{F}_t, t \geq 0\}$, we say that $\{S_t, t \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$ if and only if:

$$E[S_t \mid \mathcal{F}_s] = S_s \quad , \forall t, s$$



In discrete time, a process $\{M_n, n \in \mathbb{N}\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n \in \mathbb{N}\}$ iif $E[M_n \mid \mathcal{F}_m] = M_m$, $m \leq n$

Let's go back to the binomial:

non-arbitrage: $d \leq 1+R \leq u$

$\Rightarrow \exists\, \alpha \in [0,1]$: $1+R = \alpha u + (1-\alpha) d$

... $\Rightarrow$ new probability : $Q = \left(\frac{1+R-d}{u-d} ; \frac{u-(1+R)}{u-d} \right)$

ie : $S_1 = \begin{cases} u S_0 & \text{with prob. } \frac{1+R-d}{u-d} \\ d S_0 & \text{with prob } \frac{u-(1+R)}{u-d} \end{cases}$

$$E^{P}(S_1) = u S_0 \times p + d S_0 \times (1-p) =$$
$$= dS_0 + (u-d) S_0 p$$

$$E^{Q}(S_1) = \left[u \times \left(\frac{1+R-d}{u-d} \right) + d \times \left(\frac{u-(1+R)}{u-d} \right) \right] S_0$$

$$= \frac{u + uR - \cancel{ud} + \cancel{ud} - d - dR}{u-d} S_0$$

$$= \frac{u-d + R(u-d)}{u-d} S_0$$

$$= (1+R) S_0$$

i.e.:

$$E^{Q}\left[\frac{S_1}{1+R} \right] = S_0$$

$$\Rightarrow \quad E^{Q}\left[\frac{S_2}{1+R} \right] = S_1 \quad \ldots$$

If we use the probability $(p, 1-p)$ $\{S_n, n \in \mathbb{N}\}$?

If we use the probability $Q = \left(\frac{1+R-d}{u-d}, \frac{u-(1+R)}{u-d} \right)$,

$\left\{ \frac{S_n}{(1+R)^n}, n \in \mathbb{N} \right\}$ is a <u>martingale</u>

<u>Extension</u>:

$$E[S_t \mid \mathcal{F}_s] \leq S_s, \quad \forall s, t$$

$\{S_t, t \geq 0\}$ it is a supermartingale

$$E[S_t \mid \mathcal{F}_s] \geq S_s, \quad \forall s, t$$

$\{S_t, t \geq 0\}$ is a submartingale

<u>Property</u>: $\{M_n, n \in \mathbb{N}\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n \in \mathbb{N}\}$ iif

$$E[M_n \mid \mathcal{F}_{n-1}] = M_{n-1}, \quad \forall n$$

<u>Proof</u>: [$E[M_n \mid \mathcal{F}_m] = M_m, \ \forall m \leq n$, by definition of martingale]

($\Rightarrow$) Trivial: if it is a martingale, then in particular the property holds for $m = n-1$.

($\Leftarrow$) if $E[M_n \mid \mathcal{F}_{n-1}] = M_{n-1} \Rightarrow E[M_n \mid \mathcal{F}_m] = M_m$

$$E[\underbrace{E[M_n \mid \mathcal{F}_{n-1}]}_{M_{n-1}} \mid \mathcal{F}_{n-2}] = E[M_{n-1} \mid \mathcal{F}_{n-2}]$$

$$= E[\underbrace{E[M_{n-1} \mid \mathcal{F}_{n-2}]}_{M_{n-2}} \mid \mathcal{F}_{n-3}] = E[M_{n-2} \mid \mathcal{F}_{n-3}]$$

$$= \dots = E[M_{n-k+1} \mid \mathcal{F}_{n-k}] = M_{n-k}, \quad \forall k$$

in particular this holds for $n-k = m$ $\blacksquare$

. Let $\{\epsilon_j, j \in \mathbb{N}_0\}$ be a sequence of i.i.d. Bernoulli random variables, with $P(\epsilon_i = 1) = 0.5$. Define another sequence of random variables as follows:

$$Z_0 = 1;$$
$$Z_n = 2\epsilon_n Z_{n-1}, \quad n \geq 1$$

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a) Prove that the sequence $\{Z_n, n \geq 0\}$ is a martingale with respect to the usual filtration.

$E[\epsilon_j \epsilon_i] = E[\epsilon_j] E[\epsilon_i]$ if they are independent

$\epsilon_j \sim Ber(0.5) \qquad P(\epsilon_j = 1) = P(\epsilon_j = 0) = 0.5$

ie, we want to prove that

$$E[Z_{n+1} \mid \mathcal{F}_n^Z] = Z_n, \quad \forall n \geq 1$$

$$E[Z_{n+1} \mid \mathcal{F}_n^Z] = E[2\epsilon_{n+1} Z_n \mid \mathcal{F}_n^Z]$$

$$= 2 Z_n \, E[\underbrace{\varepsilon_{n+1}}_{\substack{\text{does not depend on } \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} \\ \Rightarrow \text{ does not depend on } \mathcal{F}_n^Z}} \mid \mathcal{F}_n^Z]$$

is $\mathcal{F}_n^Z$ - measurable

$$= 2 Z_n \, E[\varepsilon_{n+1}] = 2 Z_n (0 \times 0.5 + 1 \times 0.5) = Z_n.$$

Another way:

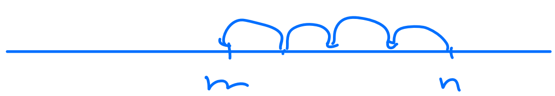
$$E[Z_n \mid \mathcal{F}_m^Z] = Z_m \quad , \forall \, m \le n-1$$

$$E[Z_n \mid \mathcal{F}_m^Z] = E[2 \varepsilon_n Z_{n-1} \mid \mathcal{F}_m^Z]$$

$$= 2 E[\varepsilon_n] E[Z_{n-1} \mid \mathcal{F}_m^Z] = E[Z_{n-1} \mid \mathcal{F}_m^Z]$$

$$= E[2 \varepsilon_{n-1} Z_{n-2} \mid \mathcal{F}_m^Z] = 2 E[\varepsilon_{n-1}] E[Z_{n-2} \mid \mathcal{F}_m^Z]$$

$$= E[Z_{n-2} \mid \mathcal{F}_m^Z] = \dots = E[Z_m \mid \mathcal{F}_m^Z] = Z_m$$



As before but with $\{\varepsilon_j, j \in \mathbb{N}\}$ being a <u>predictable sequence</u>

$$E[\varepsilon_n \mid \mathcal{F}_n^Z] = \varepsilon_n$$

$$E[\varepsilon_n \mid \mathcal{F}_{n-1}^Z] = \varepsilon_n \quad \text{(predictable)}$$

So let's see if the process $\{Z_n, n \in \mathbb{N}\}$ is still a martingale, i.e., we need to check if

$$E[Z_{n+1} \mid \mathcal{F}_n^Z] = Z_n, \quad \forall n$$

$$= E[2\,\varepsilon_{n+1} Z_n \mid \mathcal{F}_n^Z] =$$

$$= 2\,\varepsilon_{n+1}\, E(Z_n \mid \mathcal{F}_n^Z] = 2\,\varepsilon_{n+1} Z_n \neq Z_n$$

Therefore the process is <u>not</u> a martingale.

Remark: if $\varepsilon_{n+1} > 0.5 \Rightarrow 2\varepsilon_{n+1} Z_n > Z_n$
$\Rightarrow$ submartingale $\quad \varepsilon_{n+1} \in \{0.7, 1\}$

if $\varepsilon_{n+1} < 0.5 \Rightarrow 2\varepsilon_{n+1} Z_n < Z_n$
$\Rightarrow$ supermartingale

2. Assume that on the probability space $(\Omega, \mathcal{F}, P)$ there is a filtration $\{\mathcal{F}_t, t \in \mathbb{N}\}$, which is an indexed family of σ-algebras on Ω, such that $\mathcal{F}_t \subseteq \mathcal{F}$, $\forall t$. Now define the process $Y = \{Y_t, t \in \mathbb{N}\}$ by:

$$Y_t = \mathbb{E}[X|\mathcal{F}_t] \quad \in \mathcal{F}_\tau\text{-measurable}$$

Prove that $\{Y_t, t \geq 0\}$ is a $(\mathcal{F}, P)$-martingale. What can you say about any martingale M defined on a compact interval $[0, T]$, with $T < \infty$?

We need to check that $E[Y_\tau \mid \mathcal{F}_s] = Y_s, \quad \forall s \leq \tau$

$$E[\underbrace{E[X \mid \mathcal{F}_\tau]}_{Y_\tau} \mid \mathcal{F}_s] \underset{\text{tower prop.}}{=} E[\underbrace{E[X \mid \mathcal{F}_s]} \mid \mathcal{F}_\tau]$$

$$= E[Y_s \mid \mathcal{F}_\tau] = Y_s \quad (\text{because } \mathcal{F}_s \subseteq \mathcal{F}_\tau)$$

$s \leq \tau \qquad Y_s$ is $\mathcal{F}_s$ measurable $\Rightarrow$

Thus it is a martingale.